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THE STABILITY OF THE EQUILIBRIUM POSITION OF A NON-AUTONOMOUS MECHANICAL SYSTEM†

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The sufficient conditions for the asymptotic stability and instability of the equilibrium position of a holonomic mechanical system acted upon by the time-dependent forces are determined. Problems of the stabilization of the calculated motion of a gyroscopic system on a moving base and of the conditions of stability of the equilibrium position of a mechanical system with variable masses are solved. Some examples are considered. Copyright © 1996 Elsevier Science Ltd.

1. Consider a holonomic mechanical system with time-independent constraints, the position of which is defined by the general coordinates $q \in R^n$. The kinetic energy of the system $2T = \dot{q}'A(q)\dot{q}$, where the vector $\dot{q} = dq/dt$ is denoted as a column vector, $A(q)$ is an $n \times n$ matrix, positive definite for all $q \in R^n$, so that we have the matrix inequality $A(q) \geqslant A = a_0E, a_0 = \text{const} > 0$, and E is the identity matrix. Henceforth a prime denotes transposition, $\| q \|$ is the norm in R^n and $\| q \|^2 = q'q = q_1^2 + q_2^2 + \ldots + q_n^2$.

We will assume that quasipotential forces Q_1, gyroscopic forces Q_2, and dissipative-accelerating forces Q_3 act on the system, as given by

$$Q_1 = -g(t,q)\frac{\partial \Pi(q)}{\partial q}; \quad Q_2 = G(t,q,\dot{q})\dot{q}, \quad G' = -G; \quad Q_3 = -F(t,q,\dot{q})\dot{q}, \quad F' = F$$

where G and F are $n \times n$ matrices, $g, \Pi \in C^1$ are scalar non-negative functions, and $\partial\Pi/\partial q = (\partial\Pi/\partial q_1, \ldots, \partial\Pi/\partial q_n)'$.

The motion of the system can be described by Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = -g\frac{\partial \Pi}{\partial q} + G\dot{q} - F\dot{q} \tag{1.1}$$

Suppose $\partial\Pi/\partial q = 0$ when $q = 0$ and means that the system has zero position of equilibrium $\dot{q} = q = 0$. We will consider the problem of investigating the stability $\dot{q} = q = 0$ using general theorems of asymptotic stability and instability for ordinary differential equations from [1].

Solving Eqs (1.1) for $\ddot{q}$, they can be represented in the form

$$\frac{dq}{dt} = \dot{q}, \quad \frac{d\dot{q}}{dt} = \{\dot{q}'B\dot{q}\} - gA^{-1}\frac{\partial \Pi}{\partial q} + A^{-1}G\dot{q} - A^{-1}F\dot{q} \tag{1.2}$$

where $\{\dot{q}'B\dot{q}\}$ is a set of n forms, quadratic in $\dot{q}$.

We will assume that all the functions on the right-hand side of Eqs (1.2) are continuous, bounded and satisfy a Lipschitz condition with respect to $\dot{q}$ and q for each $\mu > 0$ in the range $\{t \geqslant 0, \| \dot{q} \| \leqslant \mu < +\infty, \| q \| \leqslant \mu\}$. Equations (1.2) are then precompact and regular [1, 2], and the limit equations to them have the analogous form [1]

$$\frac{dq}{dt} = \dot{q}, \quad \frac{d\dot{q}}{dt} = \{\dot{q}'B\dot{q}\} - g_*A^{-1}\frac{\partial \Pi}{\partial q} + A^{-1}G_*\dot{q} - A^{-1}F_*\dot{q} \tag{1.3}$$

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where g_*, G_*, F_* are the limits to the corresponding values from (1.2), in particular [2]

$$g_*(t,q)=\frac{d}{dt}\lim_{t_k\to+\infty}\int_0^t g(t_k+\tau,q)d\tau \tag{1.4}$$

We will assume that, for all $t\in R^+$, sufficiently small $\|\dot q\|$ and $\|q\|$, the following relations are satisfied

$$0<g_0\leqslant g(t,q)\leqslant g_1,\quad \left\|\frac{\partial g}{\partial q}(t,q)\right\|\leqslant l=\text{const} \tag{1.5}$$

$$\frac{\partial g}{\partial t}(t,q)A(q)+2g(t,q)F(t,q,\dot q)\geqslant a_0E,\quad (a_0=\text{const}>0) \tag{1.6}$$

Then, for the derivative of the function $V=(\dot q'A\dot q)/(2g(t,q))+\Pi(q)-\Pi(0)$, in view of Eqs (1.1), for sufficiently small $\|q\|$ and $\|\dot q\|$ we have the limit

$$\dot V=-\left(\frac{\partial g}{\partial t}+\dot q'\frac{\partial g}{\partial q}\right)(\dot q'A\dot q)/2g^2-(\dot q'F\dot q)/g\leqslant -b_0\|\dot q\|^2\leqslant 0$$

$$(b_0=\text{const}>0)$$

The set $\{\omega(\dot q)=b_0\|\dot q\|^2=0\}\equiv\{\dot q=0\}$, defined by this limit [1], contains only those solutions of the limit equations (1.3) (as follows directly from their structure), for which we have the relations

$$\dot q(t)\equiv 0,\quad q(t)=q_0=\text{const}:\quad g_*(t,q_0)\frac{\partial\Pi}{\partial q}(q_0)=0$$

But it follows from the first condition of (1.5) and definition (1.4) that for each $\mu>0$ for almost all $t\in[0,\mu]$, the function $g_*(t,g)\geqslant g_0>0$. Hence, these solutions will be the only solutions for which

$$q(t)=0,\quad q(t)=q_0=\text{const};\quad \frac{\partial\Pi}{\partial q}(q_0)=0$$

or the equilibrium positions of the initial system (1.1).

From the theorems given in [1] we have the following sufficient conditions for the stability of the equilibrium position of system (1.1).

Theorem 1.1. We will assume that

1. the function $\Pi(q)$ has a minimum when $q=0$;
2. the equilibrium position of system (1.1) $\dot q=q=0$ is isolated, $\|\partial\Pi/\partial q\|>0$ for $q\in\{0<\|q\|\leqslant\delta\}$;
3. the function $g(t,q)$ and the dissipative accelerating forces are such that relations (1.5) and (1.6) are satisfied.

Then, the equilibrium position (1.1) $\dot q=q=0$ is uniformly asymptotically stable.

Theorem 1.2. If instead of conditions 1 and 2 of Theorem 1.1, the following conditions are satisfied:

1′. $\Pi(q)$ has no minimum when $q=0$;

2′. in the region $\{q: 0<\|q\|\leqslant\delta,\ \Pi(q)<\Pi(0)\}$ there are no equilibrium positions of system (1.1), then the equilibrium position (1.1) $\dot q=q=0$ is unstable.

Note. It is obvious that for the case of uniform continuity [1] it can be assumed that conditions (1.5) and (1.6) of Theorems 1.1 and 1.2 are satisfied when $\dot q=q=0$.

The results obtained previously in [3] enable us to investigate the stability of the equilibrium $\dot q=q=0$

with respect to the generalized velocities and some of the generalized coordinates $(q_1, q_2, \ldots, q_m)$ $(m \leqslant n)$ [4]. For this we will use $(q^1 = (q_1, q_2, \ldots, q_m)', q^2 = (q_{m+1}, q_{m+2}, \ldots, q_n)', \| q^1 \|^2 = q_1^2 + q_2^2 + \ldots + q_m^2, \| q^2 \|^2 = q_{m+1}^2 + q_{m+2}^2 + \ldots + q_n^2, h: R^+ \to R^+$ is a Hahn-type function [4].

We will assume that the right-hand sides of (1.2) are continuous, bounded and satisfy the Lipschitz condition with respect to $\dot{q}$ and q^1 for each $\mu > 0$ in the region $\{t \geqslant 0, \| \dot{q} \| \leqslant \mu, \| q^1 \| \leqslant \mu, 0 \leqslant \| q^2 \| < +\infty\}$. Then, Eqs (1.2) are precompact in $\dot{q}$ and q^1 with respect to the arbitrary continuous function $q^2: R^+ \to R^{n-m}$ [3].

Repeating the previous discussion we can derive the following result from Theorem 5 of [3].

Theorem 1.3. We will assume that

1. the function $\Pi(q) - \Pi(0)$ is positive definite in q^1, $\Pi(q) - \Pi(0) \geqslant h(\| q^1 \|)$, i.e. for all $q \in \Gamma_0 = \{\| q^1 \| \leqslant \delta_0, \delta_0 > 0, 0 \leqslant \| q^2 \| < +\infty\}$;
2. in the region $\Gamma_0 \cap \{q: \Pi(q) - \Pi(0) > 0\}$ system (1.1) has no equilibrium positions and for all $q \in \Gamma_0 \cap \{q: \Pi(q) - \Pi(0) = \varepsilon > 0\}$ the inequality $\| \partial\Pi/\partial q \| \geqslant \delta = \delta(\varepsilon) > 0$;
3. the function $g(t, q)$ and the dissipative-accelerating forces are such that for all $t \in R^+$, $\{\dot{q}: \| \dot{q} \| \leqslant \delta_0, \delta_0 > 0\} \times \Gamma_0$ relations (1.5) and (1.6) are satisfied.

Then, the equilibrium position (1.1) $\dot{q} = q = 0$ is uniformly asymptotically stable.

Notes. 1. When $g = g(t), \dot{g}(t) \geqslant 0$ conditions (1.5) and (1.6) are satisfied if $g(t) \leqslant g_1$ and $Q_3'\dot{q} \leqslant -b_0 \| \dot{q} \|^2$, i.e. Q_3 are the forces of complete dissipation. The result on the asymptotic stability $\dot{q} = q = 0$ with respect to q under these conditions was derived previously in [5]. When $g = g(t)$ conditions (1.5) and (1.6) take the form [6]

$$0 < g_0 \leqslant g(t) \leqslant g_1, \quad \dot{g}(t)A(q) + 2g(t)F(t,q,\dot{q}) \geqslant a_0 E$$

It can be shown that if this condition is satisfied for all $t \in R^+$, $(\dot{q}, q) \in R^2$, and also if $\partial\Pi/\partial q \neq 0$ for all $q \neq 0$ and $\Pi(q) \to +\infty$ when $\| q \| \to +\infty$, the equilibrium position of the system $q = 0$ is uniformly asymptotically stable as a whole.

2. Condition (1.6) may be satisfied in the time interval $[\alpha, \beta]$ for which $\partial g/\partial t > 0$, if, even in this interval, the forces Q_3 are accelerating forces $Q_3'\dot{q} > 0$. This characteristic can be used when solving problems of the stabilization of the equilibrium position of a controlled mechanical system for "energy pumping" when $t \in [\alpha, \beta]$.

Example 1.1. Consider a mathematical pendulum with a thread of variable length $l(t)$ undergoing angular oscillations in a uniform gravitational field with constant acceleration g_0 under the action of a viscous friction torque. Denoting the angle of deflection of the pendulum from the vertical by φ, we have the following expressions for the kinetic energy and generalized forces

$$2T = m(l^2(t)\dot{\varphi}^2 + \dot{l}^2(t)), \quad Q = -mg_0 l(t)\sin\varphi - k(t,\varphi,\dot{\varphi})l^2(t)\dot{\varphi}$$

where m is the point mass and $k(t, \varphi, \dot{\varphi})$ is the coefficient of viscosity.

The equation of the oscillations of the pendulum can be represented simply in the form of Eq. (1.1) of a system with one degree of freedom. Hence, using Theorem 1.1 we can find the following sufficient conditions for uniform asymptotic stability of the lower position of equilibrium of the pendulum $\dot{\varphi} = \varphi = 0$ (the condition for the pendulum to be undriven)

$$0 < l_0 \leqslant l(t) \leqslant l_1, \quad 3\dot{l}(t) + 2k(t,0,0)m^{-1}l(t) \geqslant l_0 > 0$$

Example 1.2. Consider a symmetrical heavy body with a fixed point, placed in a uniform gravitational field of variable intensity $g = g(t)$. The centre of gravity of the body lies on the axis of symmetry (the x axis), the mass of the body is m, the principal moments of inertia A and $B = C$, and the coordinate of the centre of gravity is $x_0 > 0$. The position of the body with respect to an inertial system of coordinates with z axis directed vertically upwards will be defined by the Euler angles θ, φ, ψ in the usual way [7]. The coordinates ψ is cyclic, by ignoring which we obtain the Routh function [7] $R = R_2 + R_1 - W$ with reduced potential energy

$$W = mg(t)x_0(1 - \Pi(\theta,\varphi)) + c^2 G(\theta,\varphi)/2$$

$$\Pi(\varphi,\theta) = 1 - \sin\theta\sin\varphi, \quad G(\theta,\varphi) = ((A\sin^2\varphi + B\cos^2\varphi)\sin^2\theta + B\cos^2\theta)^{-1}$$

where c is the cyclic constant.

We find that $\partial W/\partial\theta = \partial W/\partial\varphi = 0$ when $\theta = \pi/2$, $\varphi = \pi/2$, so that we have the steady motions $\dot{\psi} =$ const, $\dot{\theta} = 0$, $\theta = \pi/2$, $\theta = \pi/2$ for which the x axis is directed vertically upwards.

We can derive the following representations

$$\partial W / \partial\theta = g(t,\theta,\varphi)\partial\Pi / \partial\theta, \quad \partial W / \partial\varphi = g(t,\theta,\varphi)\partial\Pi / \partial\varphi$$
$$g(t,\theta,\varphi) = c^2(A-B)\sin\theta\sin\varphi G^2(\theta,\varphi) - mg(t)x_0$$

This enables us, using Theorem 1.2, to determine the moments of the dissipative forces of the form $M_\theta = -k_1(t)\dot\theta$, $M_\varphi = -k_2(t)\dot\varphi$, which stabilize these vertical steady rotations of the body. We obtain the expression $2R_2 = B(\dot\theta^2 + \dot\varphi^2)$ for the function R_2 when $\theta = \varphi = \pi/2$. Then, conditions (1.5) and (1.6) in this problem can be written in the form

$$0 < a_0 \leqslant c^2(A-B) - mg(t)A^2x_0 \leqslant a_1$$
$$2c^2(A-B)k_i(t) - A^2Bm\dot g(t)x_0 \geqslant a_0 \quad (i=1,2) \tag{1.7}$$

We can conclude from Theorem 1.2 that each steady motion $\dot\theta = \dot\varphi = 0$, $\theta = \varphi = \pi/2$, corresponding to the value of the cyclic constant c satisfying conditions (1.7), is asymptotically stable with respect to $\dot\theta$, $\dot\varphi$, θ and φ.

Theorems 1.1 and 1.3 on the stability of the equilibrium position can also be extended to the case of quasipotential forces of the form

$$Q_1(t,q) = -D(t,q)A^{-1}(q)\partial\Pi(q)/\partial q \tag{1.8}$$

where $D(t, q)$ is a symmetric $n \times n$ matrix such that for all $t \in R^+$ and sufficiently small $(q, \dot q) \in \{\| \dot q \| \leqslant \delta, \| \dot q \| \leqslant \delta, \delta > 0\}$ we have the relations

$$d_0E \leqslant D(t,q) \leqslant d_1E \quad (0 < d_0 \leqslant d_1), \quad \left\| \frac{\partial d_{ij}(t,q)}{\partial q} \right\| \leqslant d_2 = \text{const}$$

$$AD^{-1}\left(F - G - \frac{\partial D}{\partial t}D^{-1}A\right) + \left(F + G - AD^{-1}\frac{\partial D}{\partial t}\right)D^{-1}A \geqslant a_0E \tag{1.9}$$

$$(a_0 = \text{const} > 0)$$

Under these conditions, by virtue of the equations of motion for the derivative of the function $2V = \dot qAD^{-1}A\dot q + 2\Pi(q)$ for sufficiently small $\| q \|$ and $\| \dot q \|$ we can obtain the limit

$$\dot V(t,q,\dot q) \leqslant -b_0 \| \dot q \|^2 \leqslant 0 \quad (b_0 = \text{const})$$

Hence it can be shown that Theorems 1.1 and 1.3 retain their formulation for the cases of forces Q_1 of the form (1.8) with conditions (1.5) and (1.6) replaced by condition (1.9).

2. We will consider the problem of determining the control force which makes the calculated motion of the gyroscopic system on a moving base stable in the formulation used previously [8].

A gyroscopic system is defined as a system constrained by constraints that are holonomic and time-independent during motion with respect to the base, containing r symmetrical gyroscopes. Its position relative to the base is specified by n generalized coordinates $q_1, q_2, \ldots, q_n$ and r angles of the natural rotations of the gyroscopes $\varphi_1, \varphi_2, \ldots, \varphi_r$. The base of the gyroscopic system performs a specified motion with respect to inertial space. Additional holonomic time-dependent constraints ensure constant velocities of the natural rotations of the gyroscopes $\dot\varphi = \dot\varphi_0 = \text{const}$.

The equations of motion of the system, converted from Lagrange's form are [8]

$$\frac{d}{dt}\frac{\partial T_2}{\partial \dot q} - \frac{\partial T_2}{\partial q} = Q + D\dot q + \frac{\partial T_0}{\partial q} - \frac{\partial B}{\partial t} \tag{2.1}$$

where $Q = Q(t, q, \dot q)$ are generalized forces, while the other quantities occurring in the equation are found from the expression for the kinetic energy of a system in absolute motion

$$T_a = T_2 + T_1 + T_0, \quad 2T_2 = \dot q'A(q)\dot q$$

$$T_1 = B'(t, q, \dot{\varphi}_0)\dot{q}, \quad T_0 = T_0(t, q, \dot{\varphi}_0), \quad D = \frac{\partial B}{\partial q'} - \frac{\partial B'}{\partial q} = -D'$$

where A and B are the corresponding $n \times n$ matrix and $n \times 1$ column matrix. The generalized forces are found from the following conditions.

1. The equations of motion (2.1) allow of the theoretical motion

$$\dot{q} = q = 0 \tag{2.2}$$

For this it is necessary to assume that for all $t \geqslant 0$

$$Q(t, 0, 0) = \left[\frac{\partial B}{\partial t} - \frac{\partial T_0}{\partial q}\right]_{q=0} \tag{2.3}$$

2. The resultant of the generalized forces $Q(t, q, \dot{q})$ is the set of forces of the gyroscopic system $Q_c = Q_c(t, q, \dot{q})$ and the special correction forces $Q_k = Q_k(t)$, which give rise to the motion (2.2). Hence

$$Q(t, q, \dot{q}) = Q_c(t, q, \dot{q}) + Q_k(t), \tag{2.4}$$

$$Q_k(t) = \left[\frac{\partial B}{\partial t} - \frac{\partial T_0}{\partial q}\right]_{q=0} - Q_c(t, 0, 0)$$

3. The forces of the strictly gyroscopic system Q_c are a set of potential forces with force function $U = U(t, q)$ and dissipative forces $Q_d(t, q, \dot{q})$, linear with respect to q

$$Q_c(t, q, \dot{q}) = \frac{\partial U(t, q)}{\partial q} + Q_d(t, q, \dot{q}) \tag{2.5}$$

$$Q_d'(t, q, \dot{q})\dot{q} \leqslant -\dot{q}'F(t, q)\dot{q} \leqslant 0$$

4. The action of the potential forces of the strictly gyroscopic system, the correction forces and the inertial forces can be represented in the form

$$\frac{\partial U}{\partial q} + Q_k + \frac{\partial T_0}{\partial q} - \frac{\partial B}{\partial t} = -g\frac{\partial \Pi_0}{\partial q}, \quad \frac{\partial \Pi_0}{\partial q} = 0 \quad \text{for} \quad q = 0 \tag{2.6}$$

where $\Pi_0 = \Pi_0(q)$ is a certain scalar function and $g = g(t, q)$ is a scalar coefficient, which satisfies the relations

$$0 < g_0 \leqslant g(t, q) \leqslant g_1, \quad \left\| \frac{\partial g}{\partial q} \right\| \leqslant g_2 \tag{2.7}$$

Propositions 1–3 do not differ from the corresponding propositions in [8]. Proposition 4 is a particularly special one compared with the corresponding one from [8]. By assuming that conditions (2.3)–(2.7) are satisfied and also requiring that the conditions of Theorem 1.1 must be satisfied, we can determine the conditions for gyroscopic systems on a moving base to be stable, which differ from those given previously in [8] by using the Lyapunov function but with a derivative of constant sign and not a derivative of fixed sign.

We will consider this difference using the following example from [8].

Example 2.1. Consider a Foucault gyroscope with two degrees of freedom of the second kind. The symmetrical rotor of the gyroscope is placed in gymbals (floating) and has a constant natural angular velocity $\dot{\varphi}_{I0}$ relative to it, and I is the axial moment of the rotor. A right trihedron $O\xi\eta\zeta$ having an absolute angular velocity $u = (u_\xi, u_\eta, u_\zeta)$ is connected with the base of the instrument, and the velocity of the origin of coordinates is v_0. The gymbals of the gyroscope are placed in bearings, the axis of which coincides with ζ, and the axis of natural rotation of the gyroscope z is perpendicular to the axis of rotation of the gymbals ζ (situated in the plane and passing through the point 0). We will assume that the centre of gravity of the system coincides with the origin of coordinates 0, and that $Oxz\zeta$ is a trihedron of the principal axes of inertia of the gymbals. A and B are the moments of inertia of the system with respect to ζ and z, and the moment of inertia of the gymbals with respect to the third axis x is such that the moments of inertia of the system with respect to x and z are equal. We will denote by α the angle between the η axis of the gyroscope, where the positive direction of the angular velocity of the gymbals $\dot\alpha$ coincides with the positive direction of ζ. The following moments act with respect to the axis of rotation of the gymbals ζ: the moment of the forces of viscous resistance $k(\dot\alpha + \omega_\zeta)$ and the correction moment $M_\zeta^k(t)$.

We will assume that the Foucault gyroscope is placed on an object which, at a given instant, is at latitude φ and moving over the surface of the Earth along a course λ with velocity v with respect to the Earth, so that [8]

$$u_\xi = 0, \quad u_\eta = \Omega + \frac{v\cos\lambda}{R\cos\varphi}, \quad u_\zeta = 0, \quad \omega_\zeta = \frac{v}{R}\cos\lambda$$

where R is the Earth's radius and Ω is the angular velocity of the Earth. We will assume that on the basis of information on the velocity v and the course λ a correction moment $M_\zeta^k(t) = kv R^{-1}\cos\lambda$ is applied to the gymbals of the gyroscope.

Then, the equations of motion of the gyroscope admit of a particular solution in which the axis of the gyroscope z constantly indicates the direction of the world axis [8]. Using Theorem 1.1 we can find that this position will be uniformly asymptotically stable provided that

$$I\dot\varphi_{10}\Omega + \frac{I\dot\varphi_{10}v\sin\lambda}{R\cos\varphi} \geqslant k_0, \quad 2k + \frac{(v\sin\lambda/\cos\varphi)^{\cdot}\cos\varphi}{\Omega R\cos\varphi + v\sin\lambda}A \geqslant k_0 > 0 \tag{2.8}$$

Note that whereas the first inequality imposes a limit on the value of the velocity and direction of motion of the object, the second imposes a limit on the change in the velocity and the course of the object. Conditions (2.8) correspond to the actual parameters of a ship and an aircraft.

Comparing conditions (2.8) with the corresponding conditions from [8] we see that the first of the inequalities of (2.8) is identical with the corresponding condition from [8]. Instead of the second inequality of (2.8) it is required in [8] that a different relation must be satisfied, namely

$$v_{\max} < \frac{k^2 R\cos\varphi}{AI\dot\varphi_{10}}\max\left(\sqrt{\frac{AI\dot\varphi_{10}\Omega}{k^2} + I} - 1\right) \tag{2.9}$$

The second inequality of (2.8) is preferable since it must follow from (2.9) that the permissible velocity of the object must decrease as the kinetic moment of the rotor increases.

3. The motion of a holonomic mechanical system with variable masses $m_\lambda = m_\lambda(t)$ ($\lambda = 1, 2, \ldots, N$) with constraints that are independent of time and with n generalized coordinates $q = (q_1, q_2, \ldots, q_n)'$, acted upon by quasipotential, gyroscopic and dissipative forces, can be described by the equations [9]

$$\frac{d^0}{dt}\left(\frac{\partial T}{\partial \dot q}\right) - \frac{\partial T}{\partial q} = -g\frac{\partial \Pi}{\partial q} + G\dot q - F\dot q + \Psi \tag{3.1}$$

$$2T = \dot q' A(m(t), q)\dot q, \quad A' = A, \quad g = g(t, m(t), q)$$

$$\Pi = \Pi(m(t), q), \quad G = G(t, q, \dot q), \quad G' = -G$$

$$F = F(t, q, \dot q), \quad F' = F, \quad \Psi = \Psi(t, q, \dot q)$$

where $\Psi(t, q, \dot q)$ are generalized reactions due to the motion of separate and connected particles inside points of the mechanical system, separated or connected to points of these particles, and d^0/dt is the derivative for fixed masses.

We will assume that the point masses of the system do not vanish and are bounded $0 < m^0 \leqslant m_\lambda(t) \leqslant m^1$ ($\lambda = 1, 2, \ldots, N$) as a result of which, in general, we have $a_0 E \leqslant A(m(t), q) \leqslant a_1 E$ ($0 < a_0 \leqslant a_1$); for all values of m_λ within these limits $\partial\Pi(m, q)/\partial q = 0$ when $q = 0$ and $\Psi = 0$ when $\dot q = q = 0$.

Then system (3.1) has an equilibrium position $\dot{q} = q = 0$.

When the conditions $g(t, m, q) > 0$, $\Pi(m, q) \geqslant 0$ are satisfied, we can use the function $V = T/g + \Pi$ to investigate the stability of $\dot{q} = q = 0$. By virtue of the equation of motion (3.1) this function has the derivative

$$\dot{V} = \frac{1}{g}\left(\dot{m}'\frac{\partial T}{\partial m} - \dot{q}'F\dot{q} + \dot{q}'\Psi + \dot{m}'\frac{\partial \Pi}{\partial m}\right) - \frac{1}{g^2}\left(\frac{\partial g}{\partial t} + \dot{m}'\frac{\partial g}{\partial m} + \dot{q}'\frac{\partial g}{\partial q}\right)T$$

By requiring this derivative to be non-positive and by also requiring the corresponding conditions of the theorems from [1, 3] to be satisfied, we can, as previously, obtain different sufficient conditions for complete and partial asymptotic stability of the equilibrium position of system (3.1) $\dot{q} = q = 0$, for example, the following theorem.

Theorem 3.1. We will assume that the following conditions are satisfied for system 3.1.

1. the function $\Pi(m, q)$ is such that $\Pi(m, 0) = 0$, $\Pi(m, q) \geqslant h(\| q \|)$ (or $\Pi(m, q) \geqslant h)(\| q^1 \|)$; this function does not increase with respect to m when the point masses of the system change, $\dot{m}'(\partial\Pi/\partial m) \leqslant 0$;
2. there are no equilibrium positions for small ($\| q \|$) (or when $\Pi(m, q) > 0$); $\| \partial\Pi(m, q)/\partial q \| > \delta(\varepsilon) > 0$ for $\| q \| = \varepsilon > 0$ (or for all $q \in \{q\colon \Pi(m, q) \geqslant \varepsilon \geqslant 0\}$);
3. reactions do not occur when the point masses of the system change;
4. for all $t \in R^+$, small $\| \dot{q} \|$ and small $\| q \|$ (or small $\| q^1 \|$) the following relations are satisfied

$$0 < g_0 \leqslant g(t, m, q) \leqslant g_1, \quad \left\| \frac{\partial g}{\partial q}(t, m, q) \right\| \leqslant l = \text{const}$$

$$\left(\frac{\partial g}{\partial t} + \dot{m}'\frac{\partial g}{\partial m}\right)A + 2gF - g\frac{\partial A}{\partial t} \geqslant a_0 E, \quad a_0 = \text{const} > 0$$

Then the position $\dot{q} = q = 0$ of system (3.1) is uniformly asymptotically stable (respectively, uniformly asymptotically stable with respect to $\dot{q}$ and q^1).

In the same way we can investigate the stability of the motions of mechanical systems with variable masses, to which the Routh–Lyapunov method [10] can be extended.

Example 3.1. Consider the motion of a solid of variable mass $m = m(t)$ with a fixed point 0, situated in a Newtonian field of force. We will assume that the outflow and inflow of particles are such that the main axes of inertia of the body with respect to the fixed point x, y, z are fixed in the body, the sum of the moments of the reactions with respect to the fixed point is equal to zero, and the centre of inertia of the body remains on the z axis of the body all the time.

We will assume that the ζ axis of the system of coordinates $O\xi\eta\zeta$, fixed in space, is directed along the radius vector $\overrightarrow{0^*0}$ where 0^* is the centre of attraction. We will denote by $\gamma_1, \gamma_2, \gamma_3$ the direction cosines of the angles of the axis ζ in the $Oxyz$ system, A_i are the principal moments of the body with respect to the Ox, Oy and Oz axes, p, q and r are the projections of the angular velocity of the body on the Ox, Oy and Oz axes, $R = | OO^* |$; $g = m/R^2$, and z_c is the coordinate of the centre of mass of the body with respect to the Oz axis.

We will assume that, in addition to the forces of attraction, forces of viscous friction act on the body, the moment of which with respect to the point O is $M' = (M_X, M_Y, M_Z)$ with the limit

$$M_X p + M_Y q + M_Z r \leqslant -(\nu_1(t)p^2 + \nu_2(t)q^2 + \nu_3(t)r^2), \quad \nu_i(t) \geqslant 0$$

Assuming that R is sufficiently large compared with the dimensions of the body, we obtain an expression for the potential energy of the body from the corresponding expression for the constant mass [11]

$$\Pi = mqz_c\gamma_3 + 3g(2R)^{-1}((A_1 - A_3)\gamma_1^2 + (A_2 - A_3)\gamma_2^2) - mgz_c$$

With the above assumptions we obtain that the body has the following positions of equilibrium

$$p = q = r = 0, \quad \gamma_1 = \gamma_2 = 0, \quad \gamma_3 = 1 \tag{3.2}$$

for which the Oz axis of the body is directed along $O\zeta$. Here the function Π is positive definite with respect to γ_1

and γ_2 in the neighbourhood of the position $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = 1$ provided that

$$p_1 = 3gR^{-1}(A_1 - A_3) - mgz_c \geqslant \varepsilon > 0$$

$$p_2 = 3gR^{-1}(A_2 - A_3) - mgz_c \geqslant \varepsilon > 0$$

Taking into account the expression for the kinetic energy of the body $2T = A_1p^2 + A_2q^2 + A_3r^2$ we have, from Theorem 3.1 the following conditions for which the positions of the body (3.2) will be uniformly asymptotically stable

$$0 < \varepsilon \leqslant A_i(t) \leqslant A_0, \quad (p_1 / p_2) \leqslant 0$$

$$(2\nu_i(t) - A_i(t))p_2(t) + \dot{p}_2(t)A_i(t) \geqslant \beta_0 > 0 \quad (i = 1,2,3)$$

If, instead of the condition $p_2 \geqslant \varepsilon > 0$, we have the condition $p_2 \leqslant -\varepsilon < 0$, then when all the remaining conditions are satisfied the set of equilibrium positions (3.2), defining the orientation of the Oz axis of the body along 0ζ, is unstable.

4. We will investigate the problem of the asymptotic stability of the equilibrium position $\dot{q} = q = 0$ of a holonomic mechanical system (1.2) in the case of potential forces $Q_1 = -\partial\Pi(t, q)/\partial q$, assuming that $\partial\Pi(t, q)/\partial q = 0$ when $q = 0$.

Suppose $\mu > 0$ is a sufficiently small number, defined by the stability domain investigated $\Gamma_0 = \{\| \dot{q} \| \leqslant \mu, \| q \| \leqslant \mu\}$. We will determine the functions

$$\alpha(t) = \sup\left(\frac{1}{\Pi(t,q)}\frac{\partial\Pi(t,q)}{\partial t} \text{ when } \| q \| \leqslant \mu\right), \quad \beta(t) = \int_0^t \alpha(\tau)d\tau$$

and we will assume that the function $\beta(t)$ is bounded, $|\beta(t)| \leqslant \beta_0$ for all $t \in R^+$, while the dissipative forces are such that for all $t \in R^+$, $(\dot{q}, q) \in \Gamma_0$

$$\alpha(t)A(q) + 2F(t, q, \dot{q}) \geqslant a_0E, \quad a_0 = \text{const} > 0 \tag{4.1}$$

Then, we can obtain the following limit for the derivative of the function $V = \exp(-\beta(t))(T + \Pi)$

$$V = \exp(-\beta(t))(-aT - \dot{q}'F\dot{q}) \leqslant -\gamma_0 \| \dot{q} \|^2 \leqslant 0 \quad (\gamma_0 = \text{const} > 0)$$

Hence, in the same way as for Theorem 1.1, we have the following sufficient conditions for asymptotic stability when there are time-dependent potential, gyroscopic and dissipative forces acting on the body.

We will assume that

1. the potential energy is such that for small $\| q \|$ we have

$$h_1(\| q \|) \leqslant \Pi(t,q) \leqslant h_2(\| q \|), \quad \| \partial\Pi / \partial q \| \geqslant \delta(\varepsilon) > 0$$

for $\| q \| = \varepsilon > 0$;

2. relation (4.1) holds.

Then the equilibrium position $\dot{q} = q = 0$ of the system is uniformly asymptotically stable.

This result is presented in [12] for the case of a mechanical system with variable masses. Other conditions of stability with respect to $(\dot{q}, q)$ and asymptotic stability with respect to $\dot{q}$ are derived in [13, 14].

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